

Zeta-Functions and Star-Products

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Abstract

We use the definition of a star (or Moyal or twisted) product to give a phasespace definition of the ζ -function. This allows us to derive new closed expressions for the coefficients of the heat kernel in an asymptotic expansion for operators of the form $\alpha p^2 + v(q)$. For the particular case of the harmonic oscillator we furthermore find a closed form for the Green's function. We also find a relationship between star exponentials, path integrals and Wigner functions, which in a simple example gives a relation between the star exponential of the Chern-Simons action and knot invariants.

1 Introduction

The star product on a given symplectic manifold is a very powerful way to quantise a given classical theory. Consider a symplectic manifold Γ , and let $\mathcal{A}_0 = C^\infty(\Gamma)$ be the Poisson-Lie algebra of classical observables. We then define a new algebra \mathcal{A}_\hbar which is the quantum analogue of \mathcal{A}_0 , by letting

$$\mathcal{A}_\hbar = \mathcal{A}_0 \otimes \mathbb{C}[[\hbar^{-1}, \hbar]] \quad (1)$$

and introducing a star product

$$f * g = fg + O(\hbar), f * g - g * f = O(\hbar) \quad (2)$$

and the Moyal bracket

$$[f, g]_M = f * g - g * f \quad (3)$$

which we demand is a deformation of the Poisson bracket, i.e.,

$$[f, g]_M = i\hbar\{f, g\}_{\text{PB}} + O(\hbar^2) \quad (4)$$

The Moyal bracket is also required to be a Lie bracket, thus making \mathcal{A}_\hbar into a Poisson-Lie algebra. It can be proven that twisted products exist on any symplectic manifold, [1], and, moreover, that different choices of star product corresponds to different choices of operator orderings, [2]. Furthermore, a W_∞ symmetry exists relating the various choices of star product, [3], and each of the various choices specify a characteristic class in the de Rham cohomology of Γ .

For $\Gamma = \mathbb{R}^{2n}$ the unique star product is

$$f * g = f e^{\frac{1}{2}i\hbar\{\cdot, \cdot\}_{\text{PB}}} g \equiv fg + \frac{1}{2}i\hbar\{f, g\}_{\text{PB}} + O(\hbar^2) \quad (5)$$

leading to the standard Moyal bracket

$$[f, g]_M = 2if \sin\left(\frac{1}{2}\hbar\{\cdot, \cdot\}_{\text{PB}}\right) g \quad (6)$$

Thus

$$[f, g]_M = i \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)!}{4} \hbar^{2n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\partial^n f}{\partial q^{n-k} \partial p^k} \frac{\partial^n g}{\partial q^k \partial p^{n-k}} \quad (7)$$

The star product comes from the Weyl map which associates a function on phasespace to each operator A on $L^2(\mathbb{R}^n)$,

$$A \mapsto A_W(q, p) \equiv \int e^{iup - ivq} \text{Tr}(\Pi(u, v)A) \, dudv \quad (8)$$

where

$$\Pi(u, v) = e^{iu\hat{p} - iv\hat{q}} \quad (9)$$

is a translation operator on phasespace giving a ray-representation of the Euclidean group E_{2n} , see [4]. The star product and the Moyal bracket can then be obtained from

$$A_W * B_W = (AB)_W \quad (10)$$

$$[A_W, B_W]_M = (AB - BA)_W \quad (11)$$

The inverse of the Weyl map associates an operator f^W to each function f on phasespace

$$f^W = \int e^{-iup+ivq} \Pi(u, v) f(q, p) du dv dq dp \quad (12)$$

and we can then write

$$f * g = (f^W g^W)_W \quad (13)$$

The Weyl map of the density operator ρ is referred to as the Wigner function. This formalism can be extended to phasespaces which are not just \mathbb{R}^{2n} , see [5].

2 Star Exponentials and Zeta Functions

Now, consider a pure state $\rho = |\psi\rangle\langle\psi|$, we then get the standard formula for the Wigner function, $W = \rho_W$,

$$W(q, p) := \int e^{iup-ivq} \text{Tr} \Pi(u, v) \rho du dv \quad (14)$$

$$= \int \langle \psi(q + y/2) | \psi(q - y/2) \rangle e^{-iyp} \frac{dy}{(2\pi)^n} \quad (15)$$

Let $|\psi_\lambda\rangle = |\lambda\rangle$ be a complete set of eigenstates to some Hermitian operator H . We can then form the heat kernel of H

$$G_H(q, q'; \sigma) = \langle q | e^{-H\sigma} | q' \rangle = \langle q | \hat{G}_H | q' \rangle \quad (16)$$

and as is well known, G_H determines the zeta function of H ,

$$\zeta_H(s) = \frac{1}{\Gamma(s)} \int d\sigma \sigma^{s-1} \text{Tr} \hat{G}_H \quad (17)$$

This provides us with a point of contact with the Wigner-Weyl-Moyal formalism, since,

$$\begin{aligned} \text{Tr} A &= \int A_W(q, p) dq dp \\ (e^A)_W(q, p) &= \text{Exp}(A_W(q, p)) \end{aligned}$$

where Exp is the $*$ -exponential,[6],

$$\text{Exp}(f) := \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{f * f * \dots * f}_n = \sum_{n=0}^{\infty} \frac{1}{n!} f^{*n} \quad (18)$$

Thus

$$\begin{aligned} \zeta_H(s) &= \frac{1}{\Gamma(s)} \int d\sigma \sigma^{s-1} \text{Tre}^{-H\sigma} \\ &= \frac{1}{\Gamma(s)} \int d\sigma \sigma^{s-1} \int \text{Exp}(-H_W(q, p)\sigma) dq dp \end{aligned} \quad (19)$$

We want to use this formula to find some new expressions for the Schwinger-DeWitt asymptotic expansion of the heat kernel. Suppose $H = \alpha p^2 + \dots$ we can then write

$$G_H(q, q'; \sigma) = \sum_{n=0}^{\infty} e^{-\frac{(q-q')^2}{4\sigma}} a_n(q, q') \sigma^{n-d/2} \quad (20)$$

in d dimensions. This implies

$$\int G_H(q, q; \sigma) dq = \sum_{n=0}^{\infty} \int a_n(q) \sigma^{n-d/2} dq = \int \text{Exp}(-H_W(q, p)\sigma) dq dp \quad (21)$$

Now, when $H = \alpha p^2 + \dots$ then $H_W = \alpha p^2 + \dots$ where the ellipsis \dots stands for terms which are at most linear in p . We can then perform the integration over p (it is a simple Gaussian integral) to obtain

$$\int \text{Exp}(-(\alpha p^2 + \dots)\sigma) dp = (\det \alpha)^{-1/2} \pi^{d/2} \sigma^{-d/2} E(q; \sigma) \quad (22)$$

where $E(q; \sigma)$ is some function depending on the precise form for H . We thus see that the $\sigma^{-d/2}$ factor comes from a Gaussian integration in this phasespace formalism. Taylor expanding $E(q; \sigma)$ in σ will then give us a_q in the Schwinger-DeWitt expansion. Notice that such asymptotic expansions are rather easily derived with this formalism. For instance, if $H = \alpha p^\gamma + v(q)$ then we get a factor $\sigma^{-d/\gamma}$ for $\gamma > 1$ from the integration over p , and we can then find a generalised Schwinger-DeWitt expansion for such (pseudo-)differential operators.

Also interesting is an expansion in powers of \hbar . Suppose f is independent of \hbar , we then write

$$\text{Exp}(f) = \sum_{n=0}^{\infty} \hbar^{2n} \mathcal{E}_{2n}(f) \quad (23)$$

By writing down the definition of the twisted product, we see that only even powers of \hbar can occur – odd powers only become relevant if the exponent itself is \hbar -dependent. Furthermore, the \hbar^2 term, say, will receive contributions from all the twisted powers of f . More precisely, we see

$$f^{*n} = f^n - \frac{\hbar^2}{8} \sum_{k=0}^{n-2} f^k \omega_2(f, f^{n-1-k}) + O(\hbar^4) \quad (24)$$

The higher orders of \hbar will not give so simple contributions. The fourth power, for example, will receive contributions not only from $f^k \omega_4(f, f^{n-1-k})$ analogous to the \hbar^2 -terms above, but also from $\omega_2(f^k, \omega_2(f^l, f^{n-k-l}))$ and $\omega_2(f^k, f^l) \omega_2(f^m, f^{n-m-k-l})$.

From this it follows that we can write

$$\begin{aligned} \mathcal{E}_2(f) &= -\frac{1}{8} \sum_{n=2}^{\infty} \frac{1}{n!} \sum_{k=0}^{n-2} f^k \omega_2(f, f^{n-1-k}) \\ &= -\frac{1}{8} \omega_2(f, f) F_2(f) - \frac{1}{2} \tilde{\omega}_2(f, f) G_2(f) \end{aligned} \quad (25)$$

where we have used

$$\omega_2(f, f^m) = m f^{m-1} \omega_2(f, f) + m(m-1) f^{m-2} \tilde{\omega}_2(f, f) \quad (26)$$

with

$$\tilde{\omega}_2(f, f) := \frac{\partial^2 f}{\partial q^2} \left(\frac{\partial f}{\partial p} \right)^2 - 2 \frac{\partial^2 f}{\partial q \partial p} \frac{\partial f}{\partial q} \frac{\partial f}{\partial p} + \frac{\partial^2 f}{\partial p^2} \left(\frac{\partial f}{\partial q} \right)^2 \quad (27)$$

The functions F_2, G_2 turn out to be

$$F_2(f) = \sum_{n=2}^{\infty} \frac{1}{n!} \sum_{k=0}^{n-2} (n-1-k) f^{n-2} = \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{n(n-3)!} f^{n-2} \quad (28)$$

$$G_2(f) = \sum_{n=2}^{\infty} \frac{1}{n!} \sum_{k=0}^{n-2} (n-1-k)(n-2-k) f^{n-3} = \frac{1}{6} \sum_{n=2}^{\infty} \frac{5n-9}{n(n-3)!} f^{n-3} \quad (29)$$

One will be able to write in a similar fashion

$$\begin{aligned} \mathcal{E}_4(f) = & \frac{i^4}{2^4 4!} (\omega_4(f, f) F_4(f) \tilde{\omega}_4(f, f) G_4(f) + \omega_2(f, f)^2 F_{2,4}(f) + \\ & \tilde{\omega}_2(f, f)^2 G_{2,4}(f) + \omega_2(f, f) \tilde{\omega}_2(f, f) H_{2,4}(f)) \end{aligned} \quad (30)$$

for suitable functions $F_4, G_4, F_{2,4}, G_{2,4}, H_{2,4}$. Clearly, as the power of \hbar grows, these expressions become more and more cumbersome. If one would wish to go to arbitrary high powers of \hbar it would be worthwhile to find a diagrammatic expression and some “Feynman rules” for constructing \mathcal{E}_n . It is clear from the recursive way these are constructed that such “Feynman rules” must exist. We will not attempt to find these rules here, however, but merely restrict ourselves to noting that often one can assume that powers of f vanish for sufficiently high powers, or at least become very simple. An example of this could be $f = \alpha p^2 + v(q)$, if the potential is not too big, one can write $f^n \approx \alpha^n p^{2n}$ for n sufficiently large. Since this particular example is of great physical importance – it gives the Hamiltonian of a Schrödinger particle for $\alpha = 1/2m$ or the Klein-Gordon field coupled to some external field in a non-minimal way (e.g. curvature) with $\alpha = 1/2, v(q) = \frac{m^2}{2} + \xi R(q) + V(q)$ – we will write down the explicit result for \mathcal{E}_2 . From the general formula we see

$$\mathcal{E}_2(\alpha p^2 + v(q)) = -\frac{1}{2} \alpha v'' F_2 - \frac{1}{4} (2\alpha^2 v'' p^2 + \alpha v'^2) G_2 \quad (31)$$

The next term, \mathcal{E}_4 , will contain third and fourth derivative of $v(q)$, hence for $v \sim q^k$ for some positive integer k the result will simplify since the derivatives of v will vanish sooner or later. For $k = 2$, the harmonic oscillator, $v(q) = \frac{1}{2} m \omega q^2, \alpha = 1/2m$ we have already

$$\mathcal{E}_2 = -\frac{1}{4} \omega F_2 - \frac{1}{8} \left(\frac{\omega}{m} p^2 + m \omega^2 q^2 \right) G_2 \quad (32)$$

We have plotted $e^f + \mathcal{E}_2(f)$ for $f = p^2 + q^2, \hbar = 1$ in figure 1. Another interesting case is $v(q) = z/q$ corresponding to the Coulomb potential (again $\alpha = 1/2m$). Here \mathcal{E}_2 becomes

$$\mathcal{E}_2 = -\frac{1}{2m} z q^{-3} F_2 - \frac{1}{4m^2} \left(z q^{-3} p^2 + \frac{m}{2} z^2 q^{-4} \right) G_2 \quad (33)$$

and $e^f + \mathcal{E}_2(f)$ has been plotted for $f = p^2 + 1/q$ in figure 2. A Yukawa-like potential $v(q) = z \exp(-\mu q)/q$ would lead to

$$\mathcal{E}_2 = -\frac{z}{2m} e^{-\mu q} (q^{-3} + \mu q^{-2} + \frac{1}{2} \mu^2 q^{-1}) F_2 -$$

$$\frac{z}{8m^2q^4}e^{-2\mu q}\left(2p^2(q+\mu q^2+\frac{1}{2}\mu^2q^3)e^{\mu q}+mz(1+2\mu q+\mu^2q^2)\right)G_2$$

We have not plotted this, since the exponential factor will only dampen the behaviour a bit but essentially it would look the same as for $\mu = 0$.

One should note that for very large values of n , the sums defining F_2, G_2 will simplify to essentially $\sum_n f^n/n!$, thus

$$F_2(-f) = e^{-f}\mathcal{F}_2(-f) \quad G_2(-f) = e^{-f}\mathcal{G}_2(-f) \quad (34)$$

where $\mathcal{F}_2, \mathcal{G}_2$ have slower growth than the exponential, making F_2, G_2 integrable as functions of p . This will be needed for the relationship with the Schwinger-DeWitt expansion to make sense. Moreover, for $f = \alpha p^2 + v(q)$ the integral over p will give $\sigma^{-d/2}$ times a function of q by simple Gaussian integration. Once more we see that the factor $\sigma^{-d/2}$ which “shifts” the powers of σ in an asymptotic expansion of the heat kernel away from the simple Taylor series arises in a straightforward manner from the phasespace formalism upon integrating out the momentum variable.

A way to systematically compute such “quantum corrections” can be found by noting that the star exponential is a solution to the differential equation

$$\frac{\partial G}{\partial \sigma} = -f * G \quad (35)$$

the solution of which is precisely $G = \text{Exp}(-f\sigma)$. Now similarly $g = e^{-f\sigma}$ is the solution of a differential equation obtained by replacing the twisted product in the above equation by an ordinary one. Write

$$G(q, p; \sigma) = \text{Exp}(-f(q, p)\sigma) = g(q, p; \sigma)\mathcal{G}(q, p; \sigma) = e^{-f(q, p)\sigma}\mathcal{G}(q, p; \sigma) \quad (36)$$

then we have that \mathcal{G} is a solution to

$$\frac{\partial \mathcal{G}}{\partial \sigma} = g^{-1}f * (g\mathcal{G}) - f\mathcal{G} \quad (37)$$

$$= -\frac{1}{2}i\hbar g^{-1}\{f, g\mathcal{G}\}_{\text{PB}} + \frac{\hbar^2}{2^2 2!}g^{-1}\omega_2(f, g\mathcal{G}) + \dots \quad (38)$$

where we have used the definition of the twisted product. Since $g = e^{-f\sigma}$ the factor g can be pulled out of the Poisson bracket cancelling the g^{-1} factor outside. For $f = \alpha p^2 + v(q)$ we get

$$\begin{aligned} \omega_2(f, g\mathcal{G}) &= 2\alpha g \left[(\sigma^2 v'^2 - 2(\sigma - \alpha p^2 \sigma^2) v'') \mathcal{G} - \right. \\ &\quad \left. 2p\sigma v'' \frac{\partial \mathcal{G}}{\partial p} + \frac{1}{2\alpha} v'' \frac{\partial^2 \mathcal{G}}{\partial p^2} - 2\sigma v' \frac{\partial \mathcal{G}}{\partial q} + \frac{\partial^2 \mathcal{G}}{\partial q^2} \right] \end{aligned} \quad (39)$$

Supposing \mathcal{G} can be Taylor expanded in powers of \hbar ,

$$\mathcal{G} = \sum_{n=0}^{\infty} \mathcal{G}_n \hbar^n \quad (40)$$

with $\mathcal{G}_0 = 1$ we get the following recursive relation

$$\frac{\partial}{\partial \sigma} \mathcal{G}_n = - \sum_{k=1}^n \frac{i^k}{2^k k!} g^{-1} \omega_k(f, g \mathcal{G}_{n-k}) \quad (41)$$

from which we see that $\mathcal{G}_1 = 0$ as expected and that \mathcal{G}_2 becomes simply

$$\mathcal{G}_2 = -\frac{1}{8} \int_0^\sigma e^{f\sigma} \omega_2(f, e^{-f\sigma}) d\sigma \quad (42)$$

which for the chosen f can be integrated quite readily to give

$$\mathcal{G}_2 = \left(\frac{1}{8} \alpha \sigma^2 v'' - \frac{1}{12} \alpha \sigma^3 (v'^2 + 2\alpha p^2 v'') \right) e^{-f\sigma} \quad (43)$$

By construction we also have

$$\mathcal{G}_2 = e^{f\sigma} \mathcal{E}_2 \quad (44)$$

leading finally to

$$\mathcal{E}_2 = \frac{1}{12} e^{-v\sigma} \sqrt{\alpha \pi} \sigma^{3/2} (2v'' - \sigma v'^2) \quad (45)$$

In a similar way, one can find analytical expressions of the remaining \mathcal{E}_n 's.

In any case, we can use the expression for the \mathcal{E}_{2n} to find expressions for the Schwinger-DeWitt coefficients order by order in \hbar . Examples of this will be given later.

For completeness we will quickly list the formula for \mathcal{G}_4 . Straightforward computations yield

$$\mathcal{G}_4 = \int_0^\sigma \left(\frac{1}{8} e^{f\sigma} \omega_2(f, e^{-f\sigma} \mathcal{G}_2) + \frac{1}{384} e^{f\sigma} \omega_4(f, e^{-f\sigma}) \right) d\sigma \quad (46)$$

$$\begin{aligned} &= \frac{1}{480} \alpha^2 \sigma^3 v^{(4)} (5 - 15\alpha p^2 \sigma + 4\alpha^2 p^4 \sigma^2) - \frac{1}{288} \alpha^2 \sigma^6 (v'^2 + 2\alpha p^2 v'')^2 + \\ &\quad \frac{1}{240} \alpha^2 \sigma^5 (9v'^2 v'' + 18\alpha p^2 v'' + 4\alpha p^2 v' v^{(3)}) + \frac{1}{48} \alpha^2 \sigma^3 v^{(4)} - \\ &\quad \frac{1}{96} \alpha^2 \sigma^2 (5v'^2 + 4v' v^{(3)} + \alpha p^2 v^{(4)}) \end{aligned} \quad (47)$$

From this one can then compute \mathcal{E}_{2n} as

$$\mathcal{E}_{2n} = e^{-f\sigma} \mathcal{G}_{2n} \quad (48)$$

As will be clear from these simple examples, the computations are all rather elementary, allowing one rather quickly to find all the relevant \mathcal{E}_{2n} 's, and, we will see, also the coefficients in a Schwinger-DeWitt asymptotic expansion for the heat kernel.

3 Path Integrals, Wigner Functions and Localisation

We can make one more important connection. Let $|\zeta\rangle$ be a set of coherent states (i.e., for the Heisenberg algebra in n dimensions, h_n , we have $\zeta \in \mathbb{C}^n$). We then have

$$\text{Exp}(-H_W\sigma) = \int \tilde{\Pi}(\bar{\zeta}, \zeta') e^{-\int_0^\sigma \tilde{H}(\bar{\zeta}, \zeta') ds} \mathcal{D}(\bar{\zeta}, \zeta') \quad (49)$$

where

$$\tilde{O}(\bar{\zeta}, \zeta') := \frac{\langle \zeta | O | \zeta' \rangle}{\langle \zeta | \zeta' \rangle} \quad (50)$$

s is a parameter along the path in ζ -space and $\mathcal{D}(\bar{\zeta}, \zeta')$ is the functional measure

$$\mathcal{D}(\bar{\zeta}, \zeta') := \lim_{N \rightarrow \infty} \prod_{i=0}^N \frac{\langle \zeta_i | \zeta_{i+1} \rangle}{\langle \zeta_i | \zeta_i \rangle} \frac{d\zeta_i}{2\pi i} \quad (51)$$

with $\zeta_0 = \zeta$, $\zeta_N = \zeta'$. See for instance [7]. This formula can be read two ways, either the functional integrals allows one to compute the star exponential, or the star exponential allows one to define a regularised functional integral. Introducing the eigenfunctions $|\lambda\rangle$ of H , we can write $H = \sum_\lambda \lambda |\lambda\rangle \langle \lambda|$, but we can also write the heat kernel as

$$G_H = e^{-H\sigma} = \sum_\lambda e^{-\lambda\sigma} |\lambda\rangle \langle \lambda| \quad (52)$$

Thus we arrive at

$$(e^{-H\sigma})_W = \text{Exp}(-H_W(q, p)\sigma) \quad (53)$$

$$= \sum_\lambda W_\lambda e^{-\lambda\sigma} \quad (54)$$

where we have defined

$$W_\lambda = (|\lambda\rangle\langle\lambda|)_W \quad (55)$$

This allows us to “localise” a functional integral, turning it into a sum over a discrete set (the spectrum of the operator). It also says that the Weyl transform of the heat kernel is a kind of Laplace transform ($\lambda \rightarrow \sigma$) of the Wigner function.

From the heat kernel one can also compute the Green’s function $G = H^{-1}$ by simply integrating over σ ,

$$G = H^{-1} = - \int_0^\infty e^{-H\sigma} d\sigma \quad (56)$$

and this implies that we can express the Green’s function in terms of the Wigner function as (using the linearity of the Weyl transform)

$$G(q, p) = - \sum_\lambda \lambda^{-1} W_\lambda(q, p) = (H^{-1})_W := H_W^{*-1} \quad (57)$$

or

$$G(q, p) = \int_0^\infty \text{Exp}(-H_W(q, p)\sigma) d\sigma \quad (58)$$

giving us an interpretation of the integral of the star exponential.

4 Applications

Our first application of the relationship between path-integrals and star-exponentials involves quantum mechanics. Let $H = \alpha p^2 + v(q)$ be the Hamiltonian for a Schrödinger particle in $d = 1$ dimension. Then

$$\Pi(u, v) = e^{iu\hat{p} - iv\hat{q}} \quad (59)$$

is the standard translation operator, giving a ray representation of the group of translations in two dimensions (i.e. on phasespace), E_2 . We then have

$$\text{Exp}(-(\alpha p^2 + v(q))\sigma) = \int e^{ip\bar{\zeta} - iq\zeta'} e^{-\int_0^\sigma \alpha \bar{\zeta}^2 + v(\zeta') ds} \mathcal{D}(\bar{\zeta}, \zeta') \quad (60)$$

$$= e^{-(\alpha p^2 + v(q))\sigma} + \hbar^2 \mathcal{E}_2(H\sigma) + O(\hbar^4) \quad (61)$$

$$= \sum_\lambda W_\lambda e^{-\lambda\sigma} \quad (62)$$

here one can interpret σ as an inverse temperature, $\sigma = \beta$. In this case, the eigenvalues λ are the energies and the W_λ become the Wigner functions of the eigenstates, i.e., the solutions to the time-independent Schrödinger equation. For the explicit example of an harmonic oscillator we then arrive at the relations

$$\text{Exp} \left(-\frac{1}{2} \left(\frac{p^2}{m} + m\omega^2 q^2 \right) \sigma \right) = \int e^{ip\bar{\zeta} - iq\zeta'} e^{-\int_0^\sigma \left(\frac{\bar{\zeta}^2}{2m} + \frac{m\omega}{2} \zeta'^2 \right) ds} \mathcal{D}(\bar{\zeta}, \zeta') \quad (63)$$

$$= e^{-\left(\frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2 \right) \sigma} - \frac{\hbar^2}{4} \omega^2 F_2 - \frac{\hbar^2}{8} \left(\frac{\omega^2}{m} p^2 + m\omega^4 q^2 \right) G_2 + O(\hbar^4) \quad (64)$$

$$= \sum_{n=0}^{\infty} W_n(p, q) e^{-\hbar\omega(n+1/2)\sigma} \quad (65)$$

where

$$W_n(p, q) = \frac{(-1)^n}{\hbar\pi n!} e^{-\frac{1}{\hbar\omega} \left(\frac{p^2}{m} + m\omega^2 q^2 \right)} L_n \left(\frac{2}{\hbar\omega} \left(\frac{p^2}{m} + m\omega^2 q^2 \right) \right) \quad (66)$$

is the Wigner function for the harmonic oscillator, [9]. This also provides us with an explicit formula for the star exponential and hence of the path integral. The expression in terms of the Wigner function (i.e., in terms of Laguerre polynomials L_n) show that we can “localise” the functional integral on a countable set, in the same way one can “localise” a curve integral in the complex plane on a finite set of poles, the value of the integral being proportional to the sum of residues at these poles. Here we have an integral over a set of more than continuum cardinality (the set of paths in the complex plane) instead of just a continuum (a specific curve in the plane), hence the set on which we “localise” is no longer finite but countable. In general one would have the functional integral localised on a set of at most continuum cardinality (corresponding to a spectrum with continuous eigenvalues and not just discrete one).

The above relation between the star exponential and the Laguerre polynomials also imply that the Green’s function for the harmonic oscillator can be written as a mode sum

$$G(q, p) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\pi \hbar^2 \omega (n+1/2) n!} e^{-\frac{1}{\hbar\omega} \left(\frac{p^2}{m} + m\omega^2 q^2 \right)} L_n \left(\frac{2}{\hbar\omega} \left(\frac{p^2}{m} + m\omega^2 q^2 \right) \right) \quad (67)$$

This is then the phasespace Green's function of this important Hamiltonian. It turns out that we can find a closed formula for this sum. The generating function for the Laguerre polynomials is

$$\frac{1}{1-t} e^{-\frac{xt}{1-t}} = \sum_{n=0}^{\infty} \frac{t^n}{n!} L_n(x)$$

Consequently,

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1/2)n!} L_n(x) = \lim_{t \rightarrow -1} t^{-3/2} \int \frac{t^{1/2}}{1-t} e^{-\frac{xt}{1-t}} dt := \mathcal{L}(x) \quad (68)$$

from which we can see at once that

$$G(q, p) = \frac{1}{\pi \hbar} \mathcal{L}(x) e^{-\frac{1}{2}x} \quad (69)$$

$$= \frac{1}{\pi \hbar} e^{-x/2} \lim_{t \rightarrow -1} t^{-3/2} \int \frac{t^{1/2}}{1-t} e^{-\frac{xt}{1-t}} dt \quad (70)$$

with $x = \frac{2}{\hbar \omega} \left(\frac{p^2}{m} + m \omega^2 q^2 \right)$. Unfortunately, I have not been able to compute this analytically but it certainly straightforward to do so numerically.

We can also find the Schwinger-DeWitt coefficients a_n in this example. By performing the integral over p and Taylor expanding the result in σ we get at once

$$a_0 = \sqrt{2\pi m} + O(\hbar^2) \quad (71)$$

$$a_1 = -\pi m^2 \omega q^2 + O(\hbar^2) \quad (72)$$

$$a_2 = \frac{1}{4} \pi m^3 \omega^2 q^4 + O(\hbar^2) \quad (73)$$

and so on.

For an arbitrary potential in one dimension we would get

$$a_0 = \sqrt{2\pi m} + O(\hbar^2) \quad (74)$$

$$a_1 = -\sqrt{2\pi m} v(q) + O(\hbar^2) \quad (75)$$

$$a_2 = \frac{\pi m}{2} v^2 + O(\hbar^2) \quad (76)$$

The \hbar^2 terms will contain derivatives of v and we see that to lowest order in \hbar we can treat the potential as a constant. In all cases, the $O(\hbar^2)$ terms are

given by $\int \mathcal{E}_2 dp$.

Inserting our expression for \mathcal{E}_2 found above, for $\alpha p^2 + v(q)$, we can readily compute the momentum integral

$$\int \mathcal{E}_2 dp = \frac{1}{12} \sqrt{\alpha\pi} \sigma^{3/2} e^{-\sigma v} (2v'' - \sigma v'^2) \quad (77)$$

Notice that the following powers of σ will appear: $3/2, 5/2, 7/2, \dots$. Thus the power $\sigma^{-1/2}$ appearing in both the phasespace Taylor series and in the asymptotic Schwinger-DeWitt expansion comes, as was to be expected, solely from the “leading symbol” p^2 , the remaining terms only giving higher powers of σ .

The $O(\hbar^2)$ correction, $\delta_2 a_n$, to the Schwinger-DeWitt coefficients then become

$$\delta_2 a_0 = \delta_2 a_1 = 0 \quad \delta_2 a_2 = \frac{1}{6} \sqrt{\alpha\pi} v'' \quad (78)$$

Similarly, by integrating \mathcal{E}_4 one can find the $O(\hbar^4)$ corrections, which will contain even higher powers of σ .

One of the problems with the standard asymptotic expansion is that it ignores boundary conditions and other global properties of the spacetime manifold and of the fields. This is a problem, for instance, in computations in curved spacetimes where one needs some such global contribution (coming from the vacuum definition) when one wants to find the renormalised energy-momentum tensor. While definitely not proven, there is some hope that the present formalism can cure some of these problems, this is so because the expressions for the heat kernel involves an integral over momentum space, i.e., on a curved manifold the cotangent bundle, and this has some information stored in it about the global properties of the spacetime manifold (how the bundle is glued together for instance). A full investigation of this problem lies beyond the scope of the present article which only claims to lay the foundation and to point out the applicability of the phasespace formalism.

As another simple example we will consider a case where the canonical variables are π_a^i, A_i^a , with $i = 1, 2, 3$ a spatial index and $a = 1, \dots, n$ a Lie algebra index, $n = \dim \mathfrak{g}$. Formally, then, the Π -function is given by

$$\Pi = e^{i\langle u, A \rangle - i\langle v, \pi \rangle} \quad (79)$$

where $u \in (\Omega^1 \otimes \mathfrak{g})^*$, $v \in \Omega^1 \otimes \mathfrak{g}$ are objects “dual” to A, π . The brackets denote the pairing between the spaces of $\{A, \pi\}$ and their duals. We can

thus write

$$\langle u, A \rangle = \int u_a^i A_i^a dx \quad \langle v, \pi \rangle = \int v_i^a \pi_a^i dx$$

We will be interested in the particular case where

$$u_a^i dx = d\gamma_a^i \quad v_i^a dx = d\eta_i^a \quad (80)$$

where then γ, η are loops. In this case Π becomes the phasespace analogue of a Wilson loop. As the observable H we will pick the Chern-Simons functional

$$H = -ig \int \text{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) dx = -iS_{\text{CS}}[A] \quad (81)$$

We then have

$$\langle \Pi \rangle_t := \int e^{i \oint_\gamma A - i \oint_\eta \pi} e^{-i \int_0^t S_{\text{CS}}[A] dt} \mathcal{D}A \mathcal{D}\pi \quad (82)$$

This path integral describes a family (indexed by t) of Chern-Simons theories. Writing

$$\oint_\eta \pi = \int \text{Tr} \pi \cdot \dot{\eta} ds = \int \text{Tr} \pi \cdot \Delta dx \quad (83)$$

where $\Delta_i^a = \delta(x, \eta(s)) \dot{\eta}_i^a$ is the so-called form factor of the loop, we can perform the π integral obtaining

$$\langle \Pi \rangle_t = \delta(\Delta(\eta)) \int e^{i \oint_\gamma A - i \int_0^t S_{\text{CS}}[A] dt} \mathcal{D}A \quad (84)$$

Two particular cases are interesting, one is where $\{A_t\}$ is a constant family, i.e., $A_t = A_0 \equiv A, \forall t$, the other is where it is a family centred on a finite number of points, i.e., $A_t \neq 0$ only for $t = t_0, \dots, t_N$. In the first case, we have $\int_0^t S_{\text{CS}} dt = t S_{\text{CS}}$ so putting $k = 4\pi g t$ and $q = \exp(\frac{2\pi i}{k+2})$ we have, according to Witten, [8],

$$\text{Exp}(-t S_{\text{CS}}) = \langle \Pi \rangle = \delta(\Delta(\eta)) c(k)^{-w(\gamma)} J_q(\gamma) \quad (85)$$

where $w(\gamma)$ is the writhe of the loop, and J_q is a Jones polynomial. This also imposes the quantisation condition $k = 4\pi g t \in \mathbb{Z}$. The second case will give a finite sum over such expressions with different coupling constants k_t and consequently indices q_t . We can consider the general expression as an interpolation between knot invariants for different values of k, q .

The equation (85) gives us an interpretation of the star exponential in this instance, and potentially also another way of computing Jones polynomials and perhaps of finding relations between them.

5 Conclusion

We found a relationship between zeta functions, path integrals, star exponentials and Wigner functions. This allowed us to regularise path integrals, derive phasespace expressions for determinants and effective actions and finally to localise path integrals in the sense of topological field theory (i.e., write a functional integral as a sum over a set of lower cardinality – either discrete or continuous). By Taylor expanding the phasespace expression for the heat kernel, we arrived at a comparison with the standard Schwinger-DeWitt expansion. We computed the Schwinger-DeWitt coefficients of the heat kernel to $O(\hbar^4)$ explicitly for $f = \alpha p^2 + v(q)$. Another explicit example was taken from topological field theory, where we constructed the star exponential of the Chern-Simons functional in three dimensions, thereby getting the Jones polynomials. The application to the heat kernel, however, seems to be the most promising.

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Figure 1: The star exponential, $\text{Exp}(f)$ to $O(\hbar^4)$, for $f = p^2 + q^2$ in units with $\hbar = 1$.

Figure 2: The star exponential for $f = p^2 + 1/q$ to $O(\hbar^4)$.



